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A finitely presented torsion-free simple group

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Abstract. We construct a finitely presented torsion-free simple group Σ_0, acting cocompactly on a product of two regular trees. An infinite family of such groups was introduced by Burger and Mozes [2], [4]. We refine their methods and construct Σ_0 as an index 4 subgroup of a group $\Sigma < \mathrm{Aut}(\mathcal{T}_{12}) \times \mathrm{Aut}(\mathcal{T}_8)$ presented by 10 generators and 24 short relations. For comparison, the smallest virtually simple group of [4, Theorem 6.4] needs more than 18000 relations, and the smallest simple group constructed in [4, §6.5] needs even more than 360000 relations in any finite presentation.

0 Introduction

Burger and Mozes constructed in [2], [4] the first examples of groups which are simultaneously finitely presented, torsion-free and simple. Moreover, they are CAT(0), bi-automatic, and have finite cohomological dimension. These groups can be realized in various ways: as fundamental groups of finite square complexes, as cocompact lattices in a product of automorphism groups of regular trees $\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$ for sufficiently large $m, n \in \mathbb{N}$, or as amalgams of finitely generated free groups. The groups of Burger and Mozes have positively answered several open questions: for example Neumann's question ([9]) on the existence of simple amalgams of finitely generated free groups, and a question of G. Mess (see [7, Problem 5.11 (C)]) on the existence of finite aspherical complexes with simple fundamental group. The construction is based on a 'normal subgroup theorem' ([4, Theorem 4.1]) which shows for a certain class of irreducible lattices acting on a product of trees, that any non-trivial normal subgroup has finite index. This statement and its remarkable proof are adapted from the famous analogous theorem of Margulis ([8, Theorem IV.4.9]) in the context of irreducible lattices in higher rank semisimple Lie groups. Another important ingredient in the construction of Burger and Mozes is a sufficient criterion ([4, Proposition 2.1]) for the non-residual-finiteness of groups acting on a product of trees. Even the bare existence of such non-residually-finite groups is remarkable, since for example finitely generated linear groups, or cocompact lattices in $\mathrm{Aut}(\mathcal{T}_k)$ are always residually finite. The non-residually-finite groups of Burger and Mozes arising from their criterion always have non-trivial normal subgroups of infinite index, but appropriate embeddings into groups satisfying the normal subgroup theorem immediately lead to virtually simple groups. Unfortunately, these groups and their simple subgroups have very large finite presentations. We therefore modify the constructions by taking a small non-residually-finite group of Wise [11, §II.5], embedding it into a group $\Sigma < \mathrm{Aut}(\mathscr{T}_{12}) \times \mathrm{Aut}(\mathscr{T}_8)$ satisfying the normal subgroup theorem, and detecting a simple subgroup $\Sigma_0 < \Sigma$ of index 4. Several GAP programs [5] have enabled us to find very quickly the groups Σ and Σ_0. The GAP code of our programs is documented in [10, Appendix B] for the interested reader.

1 Preliminaries

As mentioned in the Introduction, the finitely presented torsion-free simple groups of Burger and Mozes and of this paper appear in various forms. Probably the most comprehensible approach is to regard them as finite index subgroups of fundamental groups of certain 2-dimensional cell complexes which are called 1-vertex VH-T-square complexes in [4], complete squared VH-complexes with one vertex in [11], or $(2m, 2n)$-complexes in [10]. As in [10], we will call these fundamental groups $(2m, 2n)$-groups. Let us briefly recall their definition and some properties needed in the construction of the simple group Σ_0. Fix $m, n \in \mathbb{N}$ and let X be a finite 2-dimensional cell complex satisfying the following conditions:

- Its 1-skeleton $X^{(1)}$ consists of a single vertex x and oriented loops $a_1^{\pm 1}, \dots, a_m^{\pm 1}$, $b_1^{\pm 1}, \dots, b_n^{\pm 1}$.
- There are exactly mn geometric 2-cells attached to $X^{(1)}$. They are squares with oriented boundary of the form $aba'b'$, where $a, a' \in A := \{a_1, \dots, a_m\}^{\pm 1}$ and $b, b' \in B := \{b_1, \dots, b_n\}^{\pm 1}$. We think of the elements in A as 'horizontal' edges and the elements in B as 'vertical' edges, and do not distinguish between squares with boundary $aba'b'$, $a'b'ab$, $a^{-1}b'^{-1}a'^{-1}b^{-1}$ and $a'^{-1}b^{-1}a^{-1}b'^{-1}$, since they induce the same relations in the fundamental group of X.
- The link of the vertex x in X is the complete bipartite graph $K_{2m,2n}$ with $2m + 2n$ vertices (where the bipartite structure is induced by the decomposition $A \sqcup B$ of $X^{(1)}$ into $2m$ horizontal and $2n$ vertical edges). In other words, to any pair $(a, b) \in A \times B$ there is a uniquely determined pair $(a', b') \in A \times B$ such that $aba'b'$ is the boundary of one of the mn squares in X.

These conditions imply that the universal covering space $\tilde{X}$ of X is a product of two trees $\mathscr{T}_{2m} \times \mathscr{T}_{2n}$, where $\mathscr{T}_k$ denotes the k-regular tree. The fundamental group $\Gamma := \pi_1(X, x)$ of X is called a $(2m, 2n)$-*group*. By construction, it has a finite presentation $\Gamma = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid R_\Gamma \rangle$, where R_Γ consists of mn relations of the form $aba'b' = 1$ induced from the mn squares of X, and Γ acts freely and transitively on the vertices of $\mathscr{T}_{2m} \times \mathscr{T}_{2n}$. Moreover, it follows from the non-positive curvature of $\tilde{X}$ that Γ is torsion-free (see [1, Theorem 4.13(2)]). Equipping $\mathrm{Aut}(\mathscr{T}_k)$ with the usual topology of simple convergence and $\mathrm{Aut}(\mathscr{T}_{2m}) \times \mathrm{Aut}(\mathscr{T}_{2n})$ with the product topology, Γ can be viewed as a cocompact lattice in $\mathrm{Aut}(\mathscr{T}_{2m}) \times \mathrm{Aut}(\mathscr{T}_{2n})$. We denote by pr_1 and pr_2 the projections of Γ to the first and second factors of

$\mathrm{Aut}(\mathcal{T}_{2m}) \times \mathrm{Aut}(\mathcal{T}_{2n})$, respectively, and let H_i, $i = 1, 2$, be the closure $\overline{\mathrm{pr}_i(\Gamma)}$. Fix a vertex x_h of $\mathcal{T}_{2m}$. For each $k \in \mathbb{N}$, we can associate to a $(2m, 2n)$-group Γ a finite permutation group $P_h^{(k)}(\Gamma) < S_{2m\cdot(2m-1)^{k-1}}$ which describes the action of $\mathrm{Stab}_{H_1}(x_h)$ on the k-sphere around x_h in $\mathcal{T}_{2m}$. These 'local groups' (or at least their n generators in $S_{2m\cdot(2m-1)^{k-1}}$) can be directly computed, given the mn squares of X; see [4, Chapter 1] or [10, §1.4] for details. Similarly, one defines local vertical permutation groups $P_v^{(k)}(\Gamma) < S_{2n\cdot(2n-1)^{k-1}}$, taking the projection to the second factor $\mathrm{Aut}(\mathcal{T}_{2n})$.

There are several equivalent ways to introduce the notion of 'irreducibility' for $(2m, 2n)$-groups Γ. For example, Γ is called *irreducible* if and only if $\mathrm{pr}_2(\Gamma) < \mathrm{Aut}(\mathcal{T}_{2n})$ is not discrete. Very useful for our purposes is the following criterion of Burger and Mozes, a direct consequence of [4, Proposition 1.3] and [4, Proposition 5.2].

Proposition 1.1 (Burger and Mozes; see also [10, Proposition 1.2(1b)]). *Let Γ be a $(2m, 2n)$-group such that $n \geqslant 3$. Suppose that $P_v^{(1)}(\Gamma)$ is the alternating group A_{2n}. Then Γ is irreducible if and only if $|P_v^{(2)}(\Gamma)| = |A_{2n}| \cdot |A_{2n-1}|^{2n}$.*

Given a $(2m, 2n)$-group by its presentation $\Gamma = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \mid R_\Gamma \rangle$, we define a normal subgroup Γ_0 of index 4 as kernel of the surjective homomorphism

$$\begin{aligned} \Gamma &\to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ a_1, \ldots, a_m &\mapsto (1 + 2\mathbb{Z}, 0 + 2\mathbb{Z}), \\ b_1, \ldots, b_n &\mapsto (0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}). \end{aligned}$$

Geometrically, Γ_0 can be seen as fundamental group of a square complex with four vertices, a 4-fold regular covering of X. The subscript '0' will always refer to this specific subgroup.

We write G^* for the intersection of all finite index normal subgroups of a group G. Note that G^* is a normal subgroup of G and recall that G is called *residually finite* if and only if G^* is the trivial group. It does not matter whether one takes the intersection of all finite index subgroups or of all finite index *normal* subgroups, because of the following well-known lemma.

Lemma 1.2. *Let G be a group and H a subgroup of finite index $[G : H]$. Then there is a subgroup $N \leqslant H$ such that $N \lhd G$ and $[G : N] \leqslant [G : H]!$; in particular G^* is also the intersection of all finite index subgroups of G.*

Proof. Let $k = [G : H]$ and write G as a union of left cosets

$$G = \bigsqcup_{i=1}^{k} g_i H.$$

Left multiplication $g_iH \mapsto gg_iH$ induces a homomorphism $\phi : G \to S_k$ such that $N := \ker\phi \leqslant H$ and $[G : N] \leqslant |S_k| = [G : H]! < \infty$.

We write $\langle\langle g \rangle\rangle_G$ to denote the *normal closure* of the element $g \in G$, i.e. the intersection of all normal subgroups of G containing g.

2 The normal subgroup theorem of Burger and Mozes

Let T, T_1, T_2 be locally finite trees and let Γ be a $(2m, 2n)$-group, or more generally a subgroup of $\mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$. For $i = 1, 2$, let $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ and $H_i^{(\infty)}$ be the intersection of all closed finite index subgroups of H_i. A subgroup H of $\mathrm{Aut}(T)$ is called *locally ∞-transitive* if $\mathrm{Stab}_H(x)$ acts transitively on the k-sphere around x in T for each vertex x of T and each $k \in \mathbb{N}$.

The following statement is the general version of the normal subgroup theorem of Burger and Mozes:

Theorem 2.1 ([4, Theorem 4.1]). *Let $\Gamma < \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$ be a cocompact lattice such that H_i is locally ∞-transitive and $H_i^{(\infty)}$ is of finite index in H_i, $i = 1, 2$. Then, any non-trivial normal subgroup of Γ has finite index.*

We will use a special version of Theorem 2.1 which directly follows from the discussion in [3, Chapter 3] and [4, Chapter 5]:

Theorem 2.2 (Burger and Mozes; see also [10, Proposition 2.1]). *Let Γ be an irreducible $(2m, 2n)$-group such that $P_h^{(1)}(\Gamma)$, $P_v^{(1)}(\Gamma)$ are 2-transitive, and the stabilizers $\mathrm{Stab}_{P_h^{(1)}(\Gamma)}(\{1\})$, $\mathrm{Stab}_{P_v^{(1)}(\Gamma)}(\{1\})$ are non-abelian finite simple groups. Then any non-trivial normal subgroup of Γ has finite index.*

We can apply Theorem 2.2 for example to a group $\Lambda < \mathrm{Aut}(\mathcal{T}_6) \times \mathrm{Aut}(\mathcal{T}_6)$, acting 'locally like A_6'.

Example 2.3. Let

$$R_\Lambda := \left\{ \begin{matrix} a_1b_1a_1^{-1}b_1^{-1}, & a_1b_2a_1^{-1}b_3^{-1}, & a_1b_3a_2b_2^{-1}, \\ a_1b_3^{-1}a_3^{-1}b_2, & a_2b_1a_3^{-1}b_2^{-1}, & a_2b_2a_3^{-1}b_3^{-1}, \\ a_2b_3a_3^{-1}b_1, & a_2b_3^{-1}a_3b_2, & a_2b_1^{-1}a_3^{-1}b_1^{-1} \end{matrix} \right\}$$

and $\Lambda := \langle a_1, a_2, a_3, b_1, b_2, b_3 \mid R_\Lambda \rangle$ the corresponding $(6, 6)$-group.

Proposition 2.4. *Any non-trivial normal subgroup of Λ has finite index.*

Proof. We compute

$$P_h^{(1)}(\Lambda) = \langle (2,3)(4,5), (1,5,4,2,3), (2,3,5,4,6) \rangle \cong A_6,$$
$$P_v^{(1)}(\Lambda) = \langle (2,3)(4,5), (1,6,3,2)(4,5), (1,4,5,6)(2,3) \rangle \cong A_6,$$

and $|P_v^{(2)}(\Lambda)| = 360 \cdot 60^6$. It follows from Proposition 1.1 that Λ is irreducible. Then we apply Theorem 2.2, using that $\mathrm{Stab}_{A_6}(\{1\}) \cong A_5$ is non-abelian simple.

Computational experiments on finite index subgroups of Λ (for example using quotpic [6]) lead to the following conjecture:

Conjecture 2.5. The subgroup $\Lambda_0 < \Lambda$ is simple.

3 The simple group Σ_0

The $(8,6)$-group Δ of Example 3.1 was constructed by Wise ([11]) to give the first examples of non-residually-finite groups in the following three important classes: finitely presented small cancellation groups, automatic groups, and groups acting properly discontinuously and cocompactly on CAT(0)-spaces. We embed Δ in a $(12,8)$-group Σ such that Σ has no non-trivial normal subgroups of infinite index. The explicit knowledge of an element in Δ^* enables us to prove that the subgroup $\Sigma_0 < \Sigma$ is simple.

Example 3.1. (See [11, §II.5] where Δ is called D.) Let

$$R_\Delta := \left\{ \begin{array}{llll} a_1 b_1 a_2^{-1} b_2^{-1}, & a_1 b_2 a_1^{-1} b_1^{-1}, & a_1 b_3 a_2^{-1} b_3^{-1}, & a_1 b_3^{-1} a_2^{-1} b_2, \\ a_1 b_1^{-1} a_2^{-1} b_3, & a_2 b_2 a_2^{-1} b_1^{-1}, & a_3 b_1 a_4^{-1} b_2^{-1}, & a_3 b_2 a_3^{-1} b_1^{-1}, \\ a_3 b_3 a_4^{-1} b_3^{-1}, & a_3 b_3^{-1} a_4^{-1} b_2, & a_3 b_1^{-1} a_4^{-1} b_3, & a_4 b_2 a_4^{-1} b_1^{-1} \end{array} \right\}$$

and $\Delta := \langle a_1, a_2, a_3, a_4, b_1, b_2, b_3 \mid R_\Delta \rangle$ the corresponding $(8,6)$-group.

Proposition 3.2 ([11, [Main Theorem II.5.5]). *The group Δ is non-residually-finite and $a_2 a_1^{-1} a_3 a_4^{-1} \in \Delta^*$.*

Observe that Δ has non-trivial normal subgroups of infinite index, for example the commutator subgroup $[\Delta, \Delta]$ with infinite quotient $\Delta/[\Delta, \Delta] \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Our strategy is to embed Δ as a subgroup in a $(2m, 2n)$-group which satisfies the assumptions of Theorem 2.2, and to apply the following basic lemma.

Lemma 3.3. *Let G be a group and H a subgroup. Then $H^* \leqslant G^*$. In particular, if H is non-residually-finite, then so is G.*

Proof. Let $h \in H^*$ and $N \lhd G$ any normal subgroup of finite index. It follows that $N \cap H \lhd G \cap H = H$ such that the index $[H : (N \cap H)] \leqslant [G : N]$ is finite. Therefore $h \in N \cap H \leqslant N$.

Example 3.4. Let

$$R_\Sigma := R_\Delta \cup \left\{ \begin{array}{cccc} a_1b_4a_3b_4, & a_1b_4^{-1}a_2b_4^{-1}, & a_2b_4a_5b_4, & a_3b_4^{-1}a_4^{-1}b_4^{-1}, \\ a_4b_4^{-1}a_5b_4^{-1}, & a_5b_1a_6^{-1}b_2, & a_5b_2a_6^{-1}b_2^{-1}, & a_5b_3a_5^{-1}b_3^{-1}, \\ a_5b_2^{-1}a_6^{-1}b_1^{-1}, & a_5b_1^{-1}a_6^{-1}b_1, & a_6b_3a_6^{-1}b_4^{-1}, & a_6b_4a_6^{-1}b_3 \end{array} \right\}$$

and $\Sigma := \langle a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4 \mid R_\Sigma \rangle$ the corresponding $(12, 8)$-group.

Theorem 3.5. *The group Σ_0 is finitely presented, torsion-free and simple.*

Proof. Being a finite index subgroup of the $(12, 8)$-group Σ, the group Σ_0 is clearly finitely presented and torsion-free. It remains to prove that Σ_0 is simple.

First we show that Σ_0 has no proper subgroups of finite index. By construction, R_Σ contains all twelve elements of R_Δ. Hence by [1, Proposition II.4.14(1)], this embedding induces an injection on the level of fundamental groups, i.e. Δ is a subgroup of Σ. Let $w := a_2a_1^{-1}a_3a_4^{-1} \in \Delta < \Sigma$. By Proposition 3.2 and Lemma 3.3, Σ is non-residually-finite such that $w \in \Sigma^* \lhd \Sigma$; hence $\langle\langle w \rangle\rangle_\Sigma \leqslant \Sigma^*$ by definition of the normal closure. Coset enumeration using a computer algebra system like GAP [5] immediately shows that adding the relation $w = 1$ to the presentation of Σ leads to a finite group of order 4 (the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$); in other words, $[\Sigma : \langle\langle w \rangle\rangle_\Sigma] = 4$. It follows by definition of Σ^* that $\Sigma^* \leqslant \langle\langle w \rangle\rangle_\Sigma$; thus we have $\Sigma^* = \langle\langle w \rangle\rangle_\Sigma$. Since Σ_0 is a normal subgroup of Σ of index 4, and $w \in \Sigma_0$, we also get $\langle\langle w \rangle\rangle_\Sigma = \Sigma_0$. Now it is easy to see that the group $\Sigma_0 = \langle\langle w \rangle\rangle_\Sigma = \Sigma^*$ has no proper subgroups of finite index as follows. Assume that H is a finite index subgroup of Σ^*; then H has finite index in Σ and by Lemma 1.2 there is a finite index normal subgroup N of Σ such that $N \leqslant H \leqslant \Sigma^*$. By definition of Σ^* we have $\Sigma^* \leqslant N$, and hence $N = H = \Sigma^* = \Sigma_0$.

Next we show that Σ_0 has no non-trivial normal subgroups of infinite index. First, we observe that Σ is irreducible. This is a direct consequence of the fact that Σ is non-residually-finite, since every reducible $(2m, 2n)$-group is virtually a direct product of two free groups. Alternatively, we compute that $P_v^{(2)}(\Sigma)$ has order $20160 \cdot 2520^8$ and apply Proposition 1.1, using

$$\begin{aligned} P_v^{(1)}(\Sigma) = \langle &(1,2)(4,5)(6,8,7), (1,2,3)(4,5)(7,8), (1,2)(4,5)(6,8,7), \\ &(1,2,3)(4,5)(7,8), (1,7)(4,5), (2,8)(3,5,6,4) \rangle \cong A_8. \end{aligned}$$

We also compute

$$\begin{aligned} P_h^{(1)}(\Sigma) = \langle &(5,6)(7,8)(9,10)(11,12), (1,2)(3,4)(5,6)(7,8), \\ &(1,2)(3,4)(9,10)(11,12), (1,11,5,9,10)(2,12,3,4,8) \rangle \cong M_{12}, \end{aligned}$$

the Mathieu group which acts 5-transitively on the set $\{1, \ldots, 12\}$. Its stabilizer $\mathrm{Stab}_{M_{12}}(\{1\})$ is isomorphic to the non-abelian simple group M_{11} of order 7920. By Theorem 2.2, any non-trivial normal subgroup of Σ has finite index. Moreover,

Table 1
List of simple groups Γ^*

$P_h^{(1)}(\Gamma)$	$P_v^{(1)}(\Gamma)$	Γ/Γ^*	$\Gamma/[\Gamma,\Gamma]$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/10\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/12\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$	$=$
A_{10}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/20\mathbb{Z}$	$=$
A_{10}	A_{12}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$=$
A_{10}	A_{12}	D_6	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
A_{10}	A_{12}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$=$
A_{10}	A_{12}	$S_3\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
A_{10}	A_{12}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$=$
A_{12}	A_8	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$=$
A_{12}	A_8	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$=$
M_{12}	A_8	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$=$
A_{12}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$=$
A_{12}	A_{10}	D_6	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
A_{12}	A_{10}	$D_5\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
A_{12}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$=$
A_{12}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$=$
A_{12}	A_{10}	$D_4\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
A_{12}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$	$=$
A_{12}	A_{10}	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$	$=$

Table 1
(*Continued*)

$P_h^{(1)}(\Gamma)$	$P_v^{(1)}(\Gamma)$	Γ/Γ^*	$\Gamma/[\Gamma,\Gamma]$
A_{12}	A_{10}	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$	=
A_{12}	A_{10}	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	=
M_{12}	A_{10}	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	=
A_{12}	A_{12}	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	=
A_{12}	A_{12}	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	=
A_{12}	A_{12}	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	=

applying Theorem 2.1, any non-trivial normal subgroup of $\Sigma_0 = \Sigma^*$ has finite index. Note that one uses here again the fact that Σ^* has finite index in Σ (see the reasoning leading to [4, Corollary 5.4]).

Remark. The simple group Σ_0 has amalgam decompositions of the form $F_7 *_{F_{73}} F_7$ and $F_{11} *_{F_{81}} F_{11}$, where F_k denotes the free group of rank k. This follows from [11, Theorem I.1.18]; see also [10, Proposition 1.4]. The smallest candidate for being a finitely presented torsion-free simple group in the construction of virtually simple groups in [4, Theorem 6.4] has amalgam decompositions $F_{349} *_{F_{75865}} F_{349}$ and $F_{217} *_{F_{75601}} F_{217}$. The amalgam decompositions of the smallest simple group constructed in [4, Theorem 6.5] are $F_{7919} *_{F_{380065}} F_{7919}$ and $F_{47} *_{F_{364321}} F_{47}$.

Remark. It is easy to get an explicit finite presentation of Σ_0. Either we can take the fundamental group of the covering space corresponding to the subgroup $\Sigma_0 < \Sigma$, or we take a presentation of an amalgam mentioned in the remark above (note that its explicit construction also makes use of this covering space and additionally the Seifert–van Kampen theorem). A third possibility is to use a computer algebra system like GAP [5], and implement a Reidemeister–Schreier method. Applying this method and Tietze transformations to reduce the number of generators, we get a presentation of Σ_0 with three generators and 62 relations of total length 4866.

4 Generalization

The proof of Theorem 3.5 shows that if we embed the non-residually-finite $(8,6)$-group Δ in a $(2m,2n)$-group Γ such that $P_h^{(1)}(\Gamma)$, $P_v^{(1)}(\Gamma)$ are 2-transitive and $\mathrm{Stab}_{P_h^{(1)}(\Gamma)}(\{1\})$, $\mathrm{Stab}_{P_v^{(1)}(\Gamma)}(\{1\})$ are non-abelian simple, then the normal subgroup $\langle\langle a_2 a_1^{-1} a_3 a_4^{-1} \rangle\rangle_\Gamma$ has finite index in Γ, and $\Gamma^* = \langle\langle a_2 a_1^{-1} a_3 a_4^{-1} \rangle\rangle_\Gamma$ is a finitely presented torsion-free simple group. In this way, we have constructed many more such simple groups Γ^* for

$$(2m, 2n) \in \{(10, 10), (10, 12), (12, 8), (12, 10), (12, 12)\};$$

see Table 1. In this table, D_k denotes the dihedral group of order $2k$. Note that the index $[\Gamma : \Gamma^*]$ can be larger than 4, and that we have $[\Gamma, \Gamma] = \Gamma^*$ in most cases of Table 1.

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